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A Note on Pencils of Conics.

BY HENRY DALLAS THOMPSON.

Let the eight points in which a conic intersects a quartic be divided into two groups of four, and a conic be passed through each group: the two residual (four-point) groups lie on a conic.

Cayley's Theorem: A curve, $f_n = 0$, is cut by another, $\phi_m = 0$. The curve $\Phi_M = 0$ (where $M = m + n - \mu$, with $\mu = 1, 2, 3 \dots n$ or m), which passes through $mn - \frac{1}{2}(\mu - 1)(\mu - 2)$ of the points of intersection of $f_n = 0$ and $\phi_m = 0$, will also pass through the remaining $\frac{1}{2}(\mu - 1)(\mu - 2)$ points, *unless* these determined points be on a curve of the $(\mu - 3)^{\text{th}}$ order.*

Let $n = m = M = \mu = 4$; then, if a quartic be cut by a quartic, the quartic through thirteen of the points of intersection passes through the remaining three points.

Let $\phi_4 = 0$ and $\Phi_4 = 0$ each degenerate into a pair of conics. Then, if through a quartic (f) a conic (Φ) be passed, and through each of the two four-point groups of the points of intersection a conic (ϕ) be passed, the remaining eight points of intersection of the two latter conics (ϕ) with the quartic lie on a conic (Φ). For, through the remaining four points of the one conic and one of the remaining points of the other passes a conic (Φ). This, with the first conic (Φ), will pass through thirteen of the points of intersection of $f = 0$ and $\phi = 0$; hence, also, through the remaining three points, *unless* these three points lie on a

* Cayley's Theorem: Cambridge Math. Journal, III, 211. Cayley overlooked the exceptional case mentioned. This case was discussed and completely disposed of by Bacharach: "Ueber Schnittpunktsystem Algebraischer Curven." Inaugural Dissertation, Erlangen, 1881. Also, "Ueber den Cayley'schen Schnittpunktsatz." Math. Ann. XXVI, p. 275.

straight line ($C_{\mu-3}$); but this is impossible, as these three points lie on a conic (ϕ).*

(If one of the ϕ 's degenerates into a pair of straight lines, the four points of one of the first point-groups cannot be taken except two and two on each line. Hence the four points of the corresponding latter point-group must also lie two and two on each line, so that the theorem holds where the conics degenerate.)

The following are particular cases :

If any two conics have each double contact with a third conic, then any two of the pencil of conics through the four points of contact intersect the two tangent conics in eight points which lie on a conic.

If any two conics have each double contact with a third conic, then the eight tangents, which any two of the pencil of conics touching the four tangents of contact have in common with the first pair, touch a conic.

For the two tangent conics may be taken as the original quartic (f), and the two conics of the pencil are the cutting curve (ϕ), and through eight of the points of intersection (four of contact) passes the other conic (Φ),† leaving the remaining eight points of intersection to lie on a conic (Φ).

Let four tangents be drawn to any conic, and, at the four points of contact, let a pair of doubly tangent conics be drawn; then the eight points of intersection of these two tangent conics with the tangent lines lie on a conic.

Take any four points on a conic (or pair of lines). Let a pair of conics, tangent at these four points, be drawn; then the eight intersections of this pair of conics with a pair tangent at these four points, grouped differently, lie on a conic.

If, at each of two points on a conic, a conic be drawn with contact of the third order, and if a pencil of conics be taken touching the original conic in both of the points, then any pair of this pencil will intersect the conics with third order of contact in eight points lying on a conic.

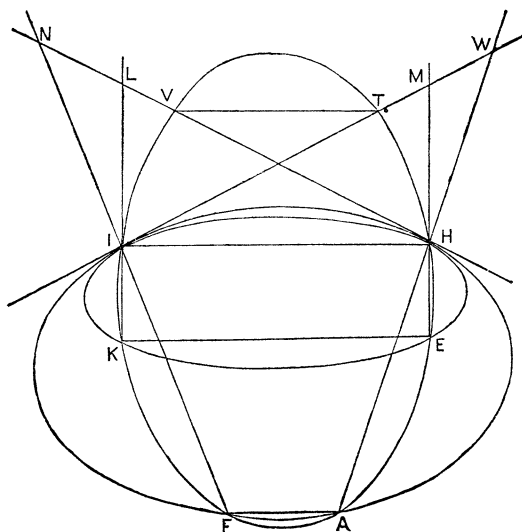
* This may be proved directly from the Riemann-Roch Theorem, $q = Q - N + \tau$, where q is the mobility of point-group, Q number of points in the group, $N = \frac{1}{2}(n-1)(n-2)$, and τ is the number of linearly independent curves of the $(n-3)$ th order through the group. In the case under consideration, $N=3$, let $q=0$ and assume $\tau=0$; hence $Q=3$. If $\tau=1$, $Q=2$.

† It is obvious that the determining conic ϕ does actually pass through eight of the points of intersection of f and ϕ . For, take the pair of conics of the pencil as each through a different four-point group on the third conic. The case considered is that when each point of one group becomes infinitely near to one of the other group.

Let any four points on a conic lie two and two on a pair of lines, and at the two points of intersection of one of these lines let a conic be taken doubly tangent to the original conic, and let another conic be taken through the four points; the last two conics will intersect again in two points collinear with the point of intersection of the two lines.

Take four tangents to any conic. Let a conic be taken tangent to the conic at two of the points of contact of tangent lines; let another conic be taken touching the four tangents; the last two conics will have two other common tangents which are concurrent with the join of the point of intersection of the two tangents taken and the point of intersection of the other two tangent lines.

For, in the figure, let IH and AF be the two lines taken; IHE the tangent conic, and $IHAFFK$ the last conic. The tangents at the points I and H to the original conic intersect the last conic in V and T . The two last conics intersect in K and E . Join H and E , H and A , E and K , F and I , K and I , V and T .



Then, by Pascal's Theorem :*

a). The polygon in the tangent conic being IK , KE , EH , tangent at H , HI , tangent at I , the join of L and M , IH and KE are concurrent.

b). The polygon in the conic $IHAFFK$ being VT , TI , IK , KE , EH and HV , the join of L and M , VT and KE are concurrent.

* *Vide Cremona : Projective Geometry, Chap. XVI.*

c). The polygon in the original conic being IF , FA , AH , tangent at H , HI , and tangent at I , the join of N and W , IH and FA are concurrent.

d). The polygon in the conic $IHAFK$ being VT , TI , IF , FA , AH and HV , the join of N and W , VT and AF are concurrent.

From a) and b) VT , IH and KE are concurrent. From c) and d) VT , IH and FA are concurrent. Hence IH , KE and FA are concurrent.

If, at the two points of intersection of the other line (FA), a doubly tangent conic be taken and another conic be passed through the four points, the eight points of intersection of the tangent conics with the two of the pencil (lying on a conic) lie two and two on four lines concurrent with the two joins taken of the original four points.

<i>Any two of the pencil of conics through four points cut the tangents at these points to a third conic of the pencil in eight points, which lie on a conic having the same Conjugate Triad with the original pencil of conics (i. e. the vertices of the quadrilateral through the four points).</i>	<i>Any two of a pencil (line) of conics touching four lines have eight tangents, which pass through the four points of contact of another one of the conics of the pencil with the four lines. These eight lines touch a conic having the same line Conjugate Triad with the original pencil.</i>
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For if successive pairs of the tangents at the points I , H , A and F (in the preceding figure) be taken as the tangent conic, the figure will be symmetrical. Hence the join of *any* two of the eight points which are taken on the same tangent conic will pass through one of the vertices of the quadrilateral $IHAF$.

The eight-point conic can degenerate only when two points from both of the cutting conics are collinear with one of the vertices of the quadrilateral through the four points of contact. Hence,

The conic through the eight points of intersection of the tangents to, with two pairs of the joins of, four points on a conic cannot degenerate.

In this latter case the eight points lie on a conic by the first extension of Pascal's theorem. For the four tangents and two pairs of joins form a polygon of eight sides in a conic, the tangents being alternate sides. Hence the external points of intersection of tangents with joins lie on a conic.